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The stability properties of a special family of parameter dependent nested hybrid multistep methods for stiff ODES

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Abstract
In this paper, we analyses the stability properties of a special family of parameter dependent nested hybrid linear multistep methods (PDNHLMMs) for the numerical integration of stiff initial value problems (IVPs) in ordinary differential equations (ODEs). Hybrid method is incorporating one or more off-step points for better stability properties. The stability properties of the method was investigated and the intervals of absolute stability of the methods of step number k ≤ 6 are determined using the boundary locus plot and the method is A- stable and
A (α) –stable which makes the methods more suitable for stiff initial value problems.

Keywords: present status, problem confrontation, oilseed cultivation, a ( )-stability

1. Introduction
This paper is concerned with the investigation and the establishment of the stability properties of the special family of a parameter dependent nested hybrid multistep method (PDNHLMM) having the general form of the *k*-step higher order and higher stage order below:

$$\begin{cases} Y_0 = \sum_{j=0}^{k} \phi_j y_{n+j} + h\lambda_k f_{n+k}; \qquad Y_0 = y_{n+c_0} \\ Y_{i+1} = y_{n+k-1} + \sum_{j=0}^{k} \varphi_j f_{n+j} + h\rho_i f(Y_i); \qquad i = 0(1)s-1; \quad Y_{i+1} = y_{n+c_{i+1}} \\ y_{n+k} = y_{n+k-1} + \sum_{j=0}^{k-2} \delta_j y_{n+j} + h\theta_{s-1} f(Y_{s-1}) + h\gamma_k (f_{n+k} + a\, f_{n+k-1}); \quad 0 \le c \le k. \end{cases} \tag{1}$$

The stability region of the formula in (1) depends on the choice of the parameter "a". It is obvious that during the computational process of the methods for the numerical integration of stiff IVPs, errors are bound to occur at some stages of the computation due to inaccuracy inherent in the formula and the arithmetic operations adopted during the computer implementation. The magnitude of the error determines the degree of accuracy and stability of the method [10]. The minimum properties a numerical method could possess are stability properties [1]. To examine the stability properties of the method (1), we give the following definitions:

Definition 1: [11]. A numerical method is said to be zero stable if the roots of the stability polynomials $\rho(r) = \sum_{j=o}^{k} \alpha_j r^j$ are inside the unit circle with simple roots on the unit circle, where $\rho(r) = \sum_{j=o}^{k} \alpha_j r^j$ is the first characteristics polynomial for the numerical method.

Definition 2: [11], a numerical method is said to be absolutely stable in a region of the complex plane if the absolute values of each of the roots of the stability polynomial is less than unity in the region.
Definition 3: [11]. A numerical method is said to be A (α)-stable if for some α ε (0, π/2) i.e. if α lies between 0 and π/2 in the region of absolute stability. The largest α (α_{max}) is regarded as the angle of absolute stability on the argument of stability.
Definition 4: [11]. A numerical method is said to be A-stable if the absolute value of the root(s) of the stability polynomial lies in the open left half of the complex plane of the stability region.
Examples of A-stable methods can be found in the work of [1, 2, 4], [5, 6], [7], [8, 9].

[8, 9] definition of stiffly stability show that stiff stability implies $A(\alpha)$-stability

2. Stability Analysis of the methods

This investigation will therefore be carried out to establish the stability properties of a special family of parameter dependent nested hybrid linear multistep methods (PDNHLMMs) by applying the method (1) for step numbers $k_{i's}$ $\forall\ i = 1(0)6$ to the

Standard scalar test equation proposed by [6]: $y' = \lambda y, \quad \mathrm{Re}(\lambda) < 0$ (2)

And substituting the values of the parameters corresponding to $k_{i's}$ gives the stability polynomial for each $k_{i's}$.

Applying the method in (1) to the test equation in (2

By setting $k = 1$ and $s = 1$ gives the stability polynomial

$$\pi_1(w,z) = w - 1 - z\theta_0(R_1(w,z)) - z\gamma_1(w + a) = 0, \quad z = \lambda h, \tag{3a}$$

Where

$$R_1(w,z) = \sum_{j=0}^{k} \hat{\alpha}_j w^j + z\beta_1 w \tag{3b}$$

Substituting the values of the parameters into (3a) and (3b) and setting $z = 0$ gives the stability polynomial for k = 1.

$$\pi_1(w) = w - 1 = 0 \tag{4}$$

The boundary locus plot of the stability polynomial for $K = 1$ is given below

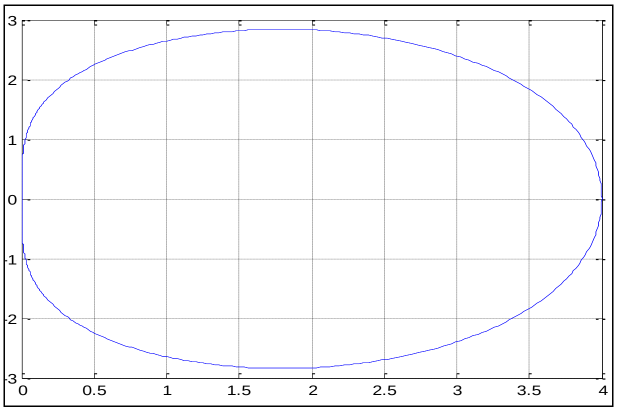

Fig 1: The stability region of the method (1) for k = 1.

The stability region is the exterior of the closed curves for k = 1 of method (1).

Again, applying the method in (1) to the scalar test equation in (2) for $k = 2$ gives the stability polynomial

$$\pi_2(w,z) = w^2 - w - z\sum_{j=0}^{0} \delta_j w^j - z\theta_1(R_2(w,z)) - z\gamma_2(w^2 + aw) = 0, \quad z = \lambda h \tag{5a}$$

Where

$$R_1(w,z) = \sum_{j=0}^{2} \phi_j w^j + z\lambda_2 w^2 \tag{5b}$$

$$R_2(w,z) = (w + z\sum_{j=0}^{2} \varphi_j w^j + z\rho_0(R_1(w,z)) \tag{5c}$$

Substituting the values of the parameters into (5a), (5b) and (5c) and setting $z = 0$ gives the stability polynomial for k = 2.

$$\pi_2(w) = w^2 - w = 0 \qquad (6)$$

Plotting the stability polynomial $\pi_2(w)$ in boundary locus sense gives the plot in fig 2.

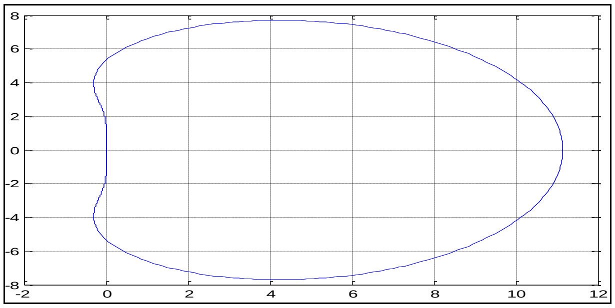

Fig 2: The stability region of the method (1) for k = 2.

The stability region is the exterior of the closed curves for k = 2 of method (1).
Again, applying the method in (1) to the test equation in (2) for $k = 3$ gives the stability polynomial

$$\pi_3(w,z) = w^3 - w^2 - z\sum_{j=0}^{1}\delta_j w^j - z\theta_2(R_3(w,z)) - z\gamma_3(w^3 + aw^2) = 0, \quad z = \lambda h \qquad (7a)$$

Where

$$R_1(w,z) = \sum_{j=o}^{3}\phi_j w^j + z\lambda_3 w^3 \qquad (7b)$$

$$R_2(w,z) = w^2 + z\sum_{j=0}^{3}\varphi_j w^j + z\rho_0(\sum_{j=o}^{3}\phi_j w^j + z\lambda_3 w^3) \qquad (7c)$$

$$R_3(w,z) = (w^2 + z\sum_{j=0}^{3}\varphi_j w^j + z\rho_1(R_2(w,z)(R_1(w,z))) \qquad (7d)$$

Substituting the values of the parameters into (7a), (7b), (7c) and (7d) and setting $z = 0$ gives the stability polynomial for k = 3.

$$\pi_3(w) = w^3 - w^2 = 0 \qquad (8)$$

Plotting the stability polynomial $\pi_3(w)$ in boundary locus sense gives the plot in fig 3.

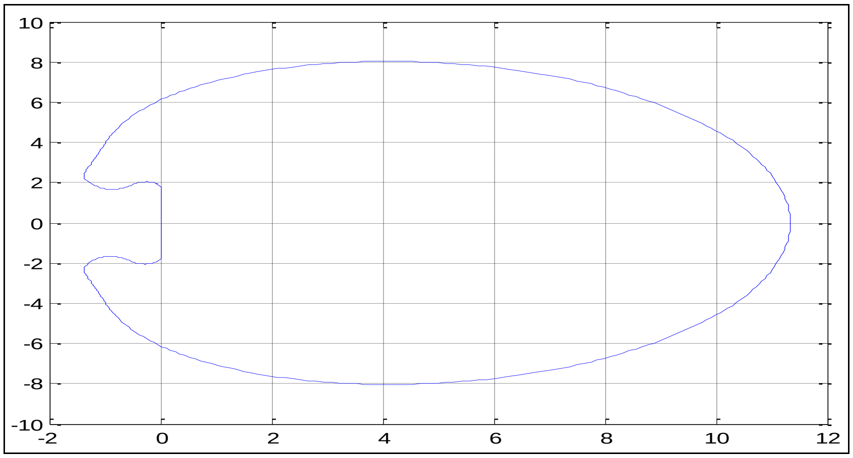

Fig 3: The stability region of the method for k = 3.

The stability region is the exterior of the closed curves for k = 3 of method (1).
Also, applying the method in (1) to the test equation in (2) for $k = 4$ gives the stability polynomial

$$\pi_4(w,z) = w^4 - w^3 - z\sum_{j=0}^{2}\delta_j w^j - z\theta_3(R_4(w,z)) + z\gamma_4(w^4 + aw^3) = 0, \quad z = \lambda h \tag{9a}$$

Where

$$R_1(w,z) = \sum_{j=0}^{4}\phi_j w^j + z\lambda_4 w^4 \tag{9b}$$

$$R_2(w,z) = w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{4}\phi_j w^j + z\lambda_4 w^4\right) \tag{9c}$$

$$R_3(w,z) = w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_1\left(w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{4}\phi_j w^j + z\lambda_4 w^4\right)\right) \tag{9d}$$

$$R_4(w,z) = (w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_2(R_3(w,z)(R_2(w,z)(R_1(w,z)))) \tag{9e}$$

Putting the values of the parameters into (9a), (9b), (9c), (9d) and (9e) and setting $z = 0$ gives the stability polynomial for k = 4.

$$\pi_4(w) = w^4 - w^3 = 0 \tag{10}$$

The boundary locus plot of the stability polynomial for $K = 4$ is shown in Fig 4 below

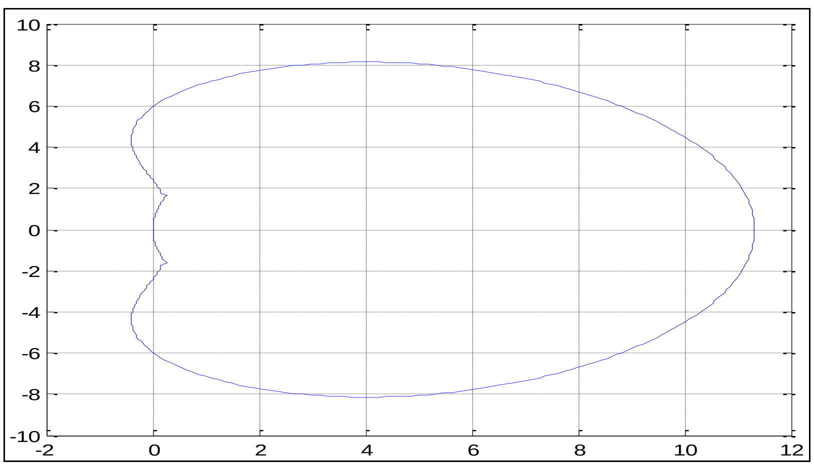

Fig 4: The stability region of the method for k = 4.

The stability region is the exterior of the closed curves for k = 4 of method (1).
Applying the method in (1) to the test equation in (2) for $k = 5$ gives the stability polynomial

$$\pi_5(w,z) = w^5 - w^4 - z\sum_{j=0}^{3}\delta_j w^j - z\theta_4(R_5(w,z)) + z\gamma_5(w^5 + aw^4) = 0, \quad z = \lambda h \tag{11a}$$

Where

$$R_1(w,z) = \sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5 \tag{11b}$$

$$R_2(w,z) = w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_0(\sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5) \tag{11c}$$

$$R_3(w,z) = w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_1(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_0(\sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5)) \tag{11d}$$

$$R_4(w,z) = w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_2(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_1(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_0(\sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5))) \tag{11e}$$

$$R_5(w,z) = (w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_3(R_4(w,z)(R_3(w,z)(R_2(w,z)(R_1(w,z)))))) \tag{11f}$$

Putting the values of the parameters into (11a), (11b), (11c), (11d), (11e) and (11f) and setting $z = 0$ gives the stability polynomial for k = 5.

$$\pi_5(w) = w^5 - w^4 = 0 \tag{12}$$

The boundary locus plot of the stability polynomial for $K = 5$ is given below in fig 5

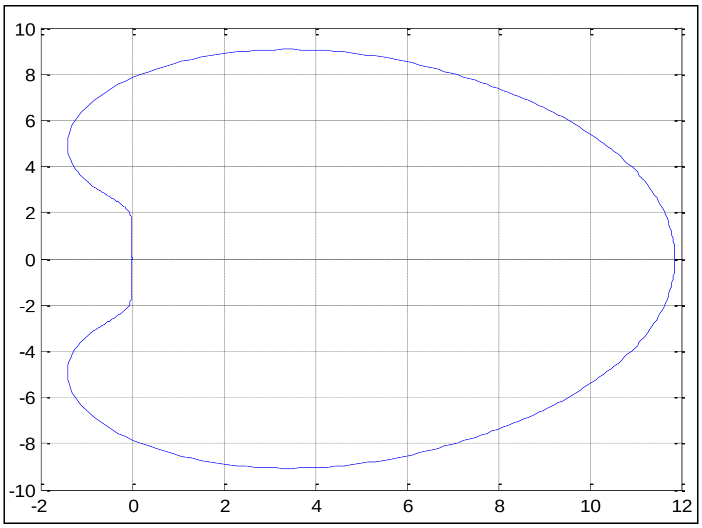

Fig 5: The stability region of the method for k = 5.

The stability region is the exterior of the closed curves for k = 5 of method (1).
Applying the method in (1) to the test equation in (2) for $k = 6$ gives the stability polynomial

$$\pi_6(w,z) = w^6 - w^5 - z\sum_{j=0}^{4}\delta_j w^j - z\theta_5(R_6(w,z)) + z\gamma_6(w^6 + aw^4) = 0, \quad z = \lambda h.. \tag{13a}$$

Where

$$R_1(w,z) = \sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6 \tag{13b}$$

$$R_2(w,z) = w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_0(\sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6) \tag{13c}$$

$$R_3(w,z) = w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_1(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_0(\sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6)) \tag{13d}$$

$$R_4(w,z) = w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_2(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_1(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_0(\sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6))) \quad (13e)$$

$$R_5(w,z) = w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_3(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_2(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_1(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_0(\sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6)))) \quad (13f)$$

$$R_6(w,z) = (w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\hat{\beta}_4(R_5(w,z)(R_4(w,z)(R_3(w,z)(R_2(w,z)(R_1(w,z)))))) \quad (13g)$$

Substituting the values of the parameters into (13a), (13b), (13c), (13d), (13e), (13f) and (4.23g) and setting $z = 0$ gives the stability polynomial for k = 6.

$$\pi_6(w) = w^6 - w^5 = 0 \quad (14)$$

Plotting the stability polynomial $\pi_6(w)$ in boundary locus sense gives the plot in fig 4.6.

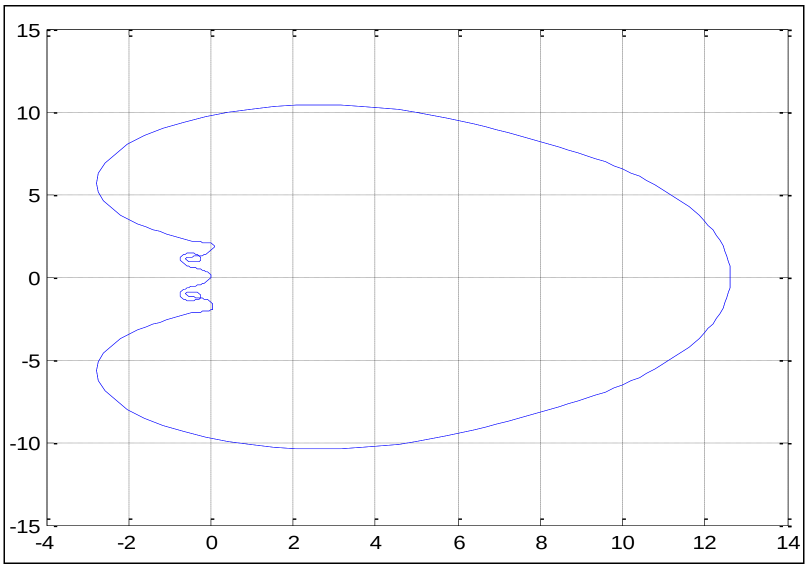

Fig 6: The stability region of the method for k = 6.

The stability region is the exterior of the closed curves for k = 6 of method (1).

Table 1: show the value of "a" and the stability properties of the method (1).

K	a	Stability properties
1	½	A-Stable (-∞, 0)
2	¼	A(α)-Stable (α = 89.9^0)
3	½	A(α)-Stable (α = 47.1^0)
4	½	A(α)-Stable (α = 82.5^0)
5	½	A(α)-Stable (α = 61.9^0)
6	¼	A(α)-Stable (α = 58.2^0)

3. Discussion of Results

The various boundary locus plots of the stability polynomial of our methods "a special family of parameter dependent nested hybrid linear multistep method (PDHNLMM)" for step number $K \leq 6$ as shown in Fig 1 to 6 clearly shows that our methods is A-stable and A(α)- stable.

4. Conclusion

Finally, it has been established from Fig 1 to 6 above that the methods "the special family of parameter dependent nested hybrid linear multistep method (PDNHLMM) is said to be A-stable and A (α)-stable in the region of absolute stability and more suitable for stiff initial value problems.

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