International Journal of Physics and Applications
Online ISSN: 2664-7583; Print ISSN: 2664-7575
Received: 07-05-2019; Accepted: 08-06-2019; Published: 09-07-2019
www.physicsjournal.in
Volume 1; Issue 1; 2019; Page No. 39-46

A special family of parameter dependent nested hybrid multistep methods for stiff ODEs

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Abstract
In this paper, we introduce a special family of parameter dependent nested hybrid linear multistep methods (PDNHLMMs) for the numerical integration of stiff initial value problems (IVPs) in ordinary differential equations (ODEs). Hybrid method is incorporating one or more off-step points for better stability properties and higher order than the conventional linear multistep methods (LMM). This type of methods provides a means of bypassing the Dahlquist order and stability barrier of the conventional LMM. The proposed method is A- stable and A (α) –stable for step number k ≤ 6. Numerical experiment shows that the new parameter dependent formulae compete favorably well with other existing methods in the literature.

Keywords: nested hybrid multi-step methods, implicit methods, A-stable, A (α)-stability

1. Introduction
Many nested formulae have been proposed for the numerical solution of stiff initial value problems in ODEs,

$$y' = f(x,y),\ x \in [x_0, X],\ y(x_0) = y_0,\ f:\Re \times \Re^m \to \Re^m \tag{1}$$

Such formulae are the adaptive nested Runge - Kutta (ANRK) methods [1], hybrid multistep formulae with nested hybrid predictors [17], second derivative linear multistep with nested predictors [16] which was introduced to overcome the limitation of order reduction that is often encountered during the implementation process in some diagonally and fully implicit Runge-Kutta methods. And also to overcome the limitation encountered by nested method in the area of long execution time when applied to large scale initial value problems. These formulae developed did very well in extensive numerical computations of stiff ODEs. The interest of this paper is to introduce a special family of parameter dependent nested hybrid multistep (PDNHLM) methods for the solution of the IVPs (1).
The general form of the *k*-step higher order and higher stage PDNHLMM is

$$\begin{cases} Y_0 = \sum_{j=0}^{k} \phi_j y_{n+j} + h\lambda_k f_{n+k}; \qquad Y_0 = y_{n+c_0} \\ Y_{i+1} = y_{n+k-1} + \sum_{j=0}^{k} \varphi_j f_{n+j} + h\rho_i f(Y_i); \qquad i = 0(1)s-1; \quad Y_{i+1} = y_{n+c_{i+1}} \\ y_{n+k} = y_{n+k-1} + \sum_{j=0}^{k-2} \delta_j y_{n+j} + h\theta_{s-1} f(Y_{s-1}) + h\gamma_k\left(f_{n+k} + a\, f_{n+k-1}\right); \quad 0 \le c \le k. \end{cases} \tag{2}$$

In (2), the parameters ϕ_j, φ_j δ_j, ρ_i, θ_{s-1}, λ_k and γ_k with $i = 1, \cdots, s-1$, $j = 0, 1, \cdots, k$ are all real coefficients. The $f_{n+j} = f\left(x_{n+j}, y_{n+j}\right)$ is the first derivative function, $h = x_{n+1} - x_n$ denotes the mesh size, k and *s* represents the step number, and the stage respectively, while $Y_i = y(x_n + c_i h) + Oh^{q_{i+1}}$, $i = 0, \cdots, s-1$, and y_{n+k} are the stage and the output point of the methods in (2). The $f(Y_i) = \{f(x_n + c_i h, Y_i)\}_{i=0}^{s-1}$ denotes the derivative of the stages. The stability region of the formulae in (2) depends on the choice of the parameter "a". The $c = [c_0, c_1, \cdots, c_s]^T$ is the abscissa vector of the input methods. The elements in c are computed from the abscissa generator: $c_i = \left\{k - \frac{1}{2^{k-i}}\right\}_{i=0}^{s-1}$, $k = 1, 2 \ldots s-1$.

2. Order conditions

By Taylor's series technique, the local truncation error of the stage method and the output method in (2) at x_n gives the following order conditions:

$$h^{q_0}:\begin{cases}\sum_{j=0}^{k}\phi_j=1, & q_0=0,\\ \sum_{j=1}^{k}j\phi_j+\lambda_k=c_0, & q_0=1,\\ \frac{1}{q_0!}\sum_{j=1}^{k}j^{q_0}\phi_j+\frac{1}{(q_0-1)!}k^{q_0-1}\lambda_k=\frac{c_0^{q_0}}{q_0\,!}; & q_0=2,3,\ldots,\end{cases} \tag{3a}$$

The error constant of the method in (2a) is

$$C_{q_0+1}=\frac{1}{(q_0+1)!}\sum_{j=0}^{k}j^{q_0+1}\phi_j+\frac{1}{q_0\,!}k^{q_0}\lambda_k=\frac{c_i^{q_0+1}}{(q_0+1)\,!}+\mathrm{O}(\mathrm{h}^{q_0+1}). \tag{3b}$$

Equations (3a) and (3b) are the order conditions and error constants of the first input method in (2) while the order conditions and error constants of the second input method in (2) are

$$h^{q_i}:\begin{cases}\sum_{j=0}^{k}\varphi_j+\rho_0=4+c_1, & i=1,2,\ldots,s,\quad i=0(1)s-1,\\ \sum_{j=1}^{k}j\varphi_j+c_0\rho_i=8-\frac{c_i^2}{2!}, & q_i=1,\\ \frac{1}{q_i\,!}\sum_{j=1}^{k}j^{q_i}\varphi_j+\rho_0c_0^{q_{i-1}}=-\frac{c_0^{q_i}}{q_i!}+\frac{64}{q_i\,!}; & q_i=2,3,\ldots,\end{cases} \tag{4a}$$

The error constant of the method in (2) is

$$C_{q_i+1}=\frac{1}{(q_i+1)!}\sum_{j=0}^{k-1}j^{q_i+1}\varphi_j+\rho_0c_0^{q_i}=\frac{c_i^{q_i+1}}{(q_i+1)\,!}+\frac{64}{(q_i+1)!}\mathrm{O}(\mathrm{h}^{q_i+1}). \tag{4b}$$

For the output method in (2), the order conditions and error constants are

$$h^{p}:\begin{cases}\sum_{j=0}^{k-1}\delta_j+\gamma_k=1, & p=0,\\ \sum_{j=1}^{k-1}\bar{j}\,\delta_j+\gamma_k(5+a)=c_4, & p=1,\\ \frac{1}{p!}\sum_{j=1}^{k-1}j^p\delta_j+\frac{1}{(p-1)\,!}\theta_{s-1}c_{s-1}^{p-1}+\frac{1}{(p-1)\,!}\left(a(-1+k)^{p-1}+k^{p-1}\right)\gamma_k=\frac{k^p}{p\,!}; & p=2,3,\ldots,\end{cases} \tag{5a}$$

And

$$C_{p+1}=\frac{1}{(p+1)!}\sum_{j=1}^{k-1}j^{p+1}\delta_j+\frac{1}{p\,!}\theta_{s-1}c_{s-1}^{p}+\frac{1}{p\,!}\left(a(-1+k)^{p}+k^{p}\right)\gamma_k=\frac{k^{p+1}}{(p+1)\,!}+\mathrm{O}(\mathrm{h}^{p+1}) \tag{5b}$$

2. Derivation of the multistep nested hybrid methods

Using the general form of the k-step, s-stage higher order PDNHLMM in (2) above, we derive the method for $k_{i's}$ by applying Taylor's series scheme to the expansion, by fixing $k_{i's}$ for every

i = 1(n)6, into (3), (4) and (5) respectively, we have by setting $k=1, s=1,\ p=q+1=3$ and $c_0=\frac{1}{2}$ yields,

$$\begin{cases} Y_0=\frac{1}{4}\left(y_n+3y_{n+1}\right)-\frac{h}{4}f_{n+1}; \quad q_0=2, \quad C_3=\frac{1}{48}, \\ y_{n+1}=y_n+\left(\frac{1-a}{2\left(\frac{-1}{2}+\frac{a}{2}\right)}\right)hf(Y_0); \ p=2, \quad C_3=\frac{1}{24}. \end{cases} \tag{6}$$

Again, for $k=2, s=2\ p=q_1,=q_0+1, c_0=\frac{7}{4}$ and $c_1=\frac{3}{2}$ in (3), (4) and (5) gives

$$Y_0=\frac{1}{256}\left(-3y_n+28y_{n+1}+231y_{n+2}\right)-\frac{21h}{128}f_{n+2}; \ C_4=\frac{7}{2048}, \ q_0=3,$$

$$Y_1=y_{n+1}+\frac{1}{96}\left(-hf_n+30hf_{n+1}-13hf_{n+2}+32h\,f(Y_0)\right); \quad C_5=\frac{179}{92160}, \quad q_1=4,$$

$$y_{n+2}=y_{n+1}+\frac{2}{9}\left(-\frac{1-a}{-2+a}hf_n-\frac{8-5a}{-2+a}hf(Y_1)\right)-\frac{1}{6(-2+a)}h\left(f_{n+2}+af_{n+1}\right); \quad C_4=\frac{1}{84}, \quad p=3. \tag{7}$$

Setting $k=3, s=3, p=q_2=q_1=q_0+1=5, c_0=\frac{23}{8}, c_1=\frac{11}{4}$ and $c_2=\frac{5}{2}$ in (3), (4) and (5) respectively gives,

$$Y_0=\frac{1}{49152}\left(70y_n-48328y_{n+1}+2070y_{n+2}+47495y_{n+3}-4830hf_{n+3}\right); \ C_5=\frac{161}{262144}, q_0=4,$$

$$Y_1=y_{n+2}+\frac{1143}{10496}hf_n-\frac{1983}{51200}hf_{n+1}+\frac{30549}{71680}hf_{n+2}-\frac{4977}{10240}hf_{n+3}+\frac{3396h}{824025}f(Y_0); \ C_6=\frac{681}{409600}, q_1=5,$$

$$Y_2=y_{n+2}+\frac{179}{63360}hf_n-\frac{319}{13440}hf_{n+1}+\frac{1979}{5760}hf_{n+2}-\frac{601}{5760}hf_{n+3}+\frac{976h}{3465}f(Y_1); \quad C_6=\frac{41}{46080}, q_2=5,$$

$$y_{n+3}=y_{n+2}-\frac{(-1+a)}{30(-3+a)}hf_n-\frac{(1-3a)}{6(-3+a)}hf_{n+1}-\frac{(19-9a)}{\frac{15}{2}(-3+a)}hf(Y_2)+\frac{h}{3(-3+a)}\left(f_{n+3}+af_{n+2}\right); C_5=\frac{11}{2880}, p=4. \tag{8}$$

For $k=4, s=4, p=q_3=q_2=q_1=q_0+1=6$, $c_0=\frac{63}{16}, c_1=\frac{23}{8}$, $c_2=\frac{15}{4}$, and $c_3=\frac{7}{2}$ in (3), (4) and (5) respectively produces,

$$Y_0=-\frac{7.3\times10^3}{3.4\times10^7}y_n+\frac{3.3\times10^3}{2.1\times10^6}y_{n+1}-\frac{4.4\times10^4}{8.4\times10^6}y_{n+2}+\frac{3.1\times10^4}{2.1\times10^6}y_{n+3}+\frac{3.3\times10^7}{3.3\times10^7}y_{n+4}-\frac{4.6\times10^5}{8.4\times10^6}hf_{n+4}; \ C_6=\frac{3.1\times10^4}{2.7\times10^8}, \ q_0=5,$$

$$Y_1=y_{n+3}-\frac{1.3\times10^5}{5.3\times10^7}hf_n+\frac{6.7\times10^5}{3.4\times10^7}hf_{n+1}-\frac{2.3\times10^6}{3.0\times10^7}hf_{n+2}+\frac{2.7\times10^4}{5.5\times10^4}hf_{n+3}-\frac{6.8\times10^6}{5.8\times10^6}hf_{n+4}+\frac{9.4\times10^5}{5.9\times10^5}hf(Y_0);$$
$$C_7=\frac{2.0\times10^7}{1.8\times10^{10}}, \ q_1=6,$$

$$Y_2 = y_{n+3} - \frac{6.8\times10^2}{3.2\times10^5}hf_n + \frac{3.9\times10^3}{2.3\times10^5}hf_{n+1} - \frac{6.7\times10^2}{1.0\times10^4}hf_{n+2} + \frac{3.3\times10^4}{7.2\times10^4}hf_{n+3} - \frac{5.4\times10^2}{1.3\times10^3}hf_{n+4} + \frac{1.9\times10^4}{2.5\times10^4}hf(Y_1);$$
$$C_7 = -\frac{3.5\times10^4}{3.7\times10^7},\ q_2 = 6,$$

$$Y_3 = y_{n+3} - \frac{41}{34560}hf_n + \frac{589}{63360}hf_{n+1} - \frac{131}{3360}hf_{n+2} + \frac{6347}{17280}hf_{n+3} - \frac{997}{11520}hf_{n+4} + \frac{520}{2079}hf(Y_2); C_7 = -\frac{3.5\times10^{-4}}{3.7\times10^{-7}}, q_3 = 6,$$

$$y_{n+4} = y_{n+3} - \frac{\frac{1}{4}(57-59a)}{630(-4+a)}hf_n - \frac{4(-232+237a)}{450(-4+a)}hf_{n+1} - \frac{4(118-119a)}{90(-4+a)}hf_{n+2} - \frac{4(646-251a)}{\frac{1575}{2}(-4+a)}hf(Y_3) - \frac{179}{360(-4+a)}h\left(f_{n+4} + a\ f_{n+3}\right); C_6 = \frac{32}{35151},\ p = 5. \tag{9}$$

For $k = 5$, $s = 5$, $p = q_4 = q_3 = q_2 = q_1 = q_0 + 1 = 7$, $c_0 = \frac{159}{32}$, $c_1 = \frac{79}{16}$, $c_2 = \frac{39}{8}$, $c_3 = \frac{19}{4}$, and $c_4 = \frac{9}{2}$ in (3), (4) and (5) respectively yields the PDNHLM methods of order seven

$$Y_0 = \frac{1.6\times10^6}{4.2\times10^{10}}y_n - \frac{9.8\times10^6}{3.4\times10^{10}}y_{n+1} + \frac{4.4\times10^6}{4.3\times10^9}y_{n+2} - \frac{2.0\times10^7}{8.6\times10^9}y_{n+3} + \frac{4.1\times10^7}{8.6\times10^9}y_{n+4} + \frac{1.7\times10^{11}}{1.7\times10^{11}}y_{n+5} - \frac{2.5\times10^8}{8.6\times10^9}hf_{n+5};$$
$$C_7 = \frac{1.2\times10^7}{5.5\times10^{11}},\quad q_0 = 6,$$

$$Y_1 = y_{n+4} + \frac{1.5\times10^8}{9.9\times10^{10}}hf_n - \frac{2.9\times10^9}{2.4\times10^{11}}hf_{n+1} + \frac{8.3\times10^8}{1.8\times10^{10}}hf_{n+2} - \frac{2.4\times10^9}{2.0\times10^{10}}hf_{n+3} + \frac{3.2\times10^{10}}{5.8\times10^{10}}hf_{n+4} - \frac{4.5\times10^9}{1.9\times10^9}hf_{n+5}$$
$$+ \frac{1.6\times10^9}{5.8\times10^8}hf(Y_0);\ C_7 = \frac{2.2\times10^{10}}{2.7\times10^{13}},\quad q_1 = 7,$$

$$Y_2 = y_{n+4} + \frac{2.1\times10^7}{1.5\times10^{10}}hf_n - \frac{1.9\times10^7}{1.7\times10^9}hf_{n+1} + \frac{1.9\times10^8}{4.4\times10^9}hf_{n+2} - \frac{3.3\times10^8}{2.9\times10^9}hf_{n+3} + \frac{3.0\times10^8}{5.7\times10^8}hf_{n+4} - \frac{1.9\times10^8}{1.9\times10^8}hf_{n+5} + \frac{6.8\times10^7}{4.7\times10^7}hf(Y_1);$$
$$C_8 = \frac{2.2\times10^8}{2.9\times10^{11}},\quad q_2 = 7,$$

$$Y_3 = y_{n+4} + \frac{3.5\times10^4}{3.0\times10^7}hf_n - \frac{6.7\times10^5}{7.1\times10^7}hf_{n+1} + \frac{9.5\times10^5}{2.6\times10^7}hf_{n+2} - \frac{2.2\times10^4}{2.3\times10^5}hf_{n+3} + \frac{8.0\times10^6}{1.6\times10^7}hf_{n+4} - \frac{8.6\times10^5}{2.3\times10^6}hf_{n+5} + \frac{1.6\times10^6}{2.3\times10^6}hf(Y_2);$$
$$C_8 = \frac{1.2\times10^6}{2.1\times10^9}hf,\quad q_3 = 7,$$

$$Y_4 = y_{n+4} + \frac{7.6\times10^2}{1.2\times10^6}hf_n - \frac{4.9\times10^3}{9.8\times10^5}hf_{n+1} + \frac{3.5\times10^4}{1.8\times10^6}hf_{n+2} - \frac{6.3\times10^4}{1.1\times10^6}hf_{n+3} + \frac{3.7\times10^5}{9.7\times10^5}hf_{n+4} - \frac{2.4\times10^4}{3.2\times10^5}hf_{n+5}$$
$$+ \frac{6.3\times10^4}{2.8\times10^5}hf(Y_3);\quad C_7 = \frac{1.3\times10^4}{4.3\times10^7},\quad q_4 = 7,$$

$$y_{n+5} = y_{n+4} - \frac{53-58a}{3240(-5+a)}hf_n + \frac{(-90+97a)}{840(-5+a)}hf_{n+1} + \frac{(37-39a)}{120(-5+a)}hf_{n+2} + \frac{589}{1080} + \frac{79}{36(-5+a)}hf_{n+3}$$
$$+ \frac{8\left(475 + \frac{948}{-5+a}\right)}{2835}hf(Y_4) - \frac{79}{120(-5+a)}h\left(f_{n+5} + a\ f_{n+4}\right);\quad C_7 = \frac{373}{362880},\quad p = 6. \tag{10}$$

Setting $k=6$, $s=6$, $p=q_5=q_4=q_3=q_2=q_1=q_0+1=8$, $c_0=\frac{383}{64}$, $c_1=\frac{191}{32}$, $c_2=\frac{95}{16}$, $c_3=\frac{47}{8}$, $c_4=\frac{23}{4}$, And $c_5=\frac{11}{2}$ in (3), (4) and (5) respectively yields the PDNHLM methods of order eight;

$$Y_0=-\frac{9.2\times10^{8}}{1.4\times10^{13}}y_n+\frac{9.9\times10^{9}}{1.7\times10^{14}}y_{n+1}-\frac{6.2\times10^{10}}{2.8\times10^{14}}y_{n+2}+\frac{9.2\times10^{9}}{1.8\times10^{13}}y_{n+3}-\frac{1.2\times10^{11}}{1.4\times10^{14}}y_{n+4}+\frac{5.0\times10^{10}}{3.5\times10^{13}}y_{n+5}$$
$$-\frac{1.4\times10^{15}}{1.4\times10^{15}}y_{n+6}-\frac{1.1\times10^{12}}{7.0\times10^{13}}hf_{n+6};\ C_8=\frac{1.5\times10^{11}}{3.6\times10^{16}},\ \ q_0=7,$$

$$Y_1=y_{n+5}-\frac{1.6\times10^{14}}{1.6\times10^{17}}hf_n+\frac{6.3\times10^{13}}{7.4\times10^{15}}hf_{n+1}-\frac{1.4\times10^{14}}{3.9\times10^{15}}hf_{n+2}+\frac{8.6\times10^{15}}{9.9\times10^{16}}hf_{n+3}-\frac{4.9\times10^{15}}{2.9\times10^{16}}hf_{n+4}+\frac{2.3\times10^{13}}{3.9\times10^{13}}hf_{n+5}-\frac{9.9\times10^{15}}{2.1\times10^{15}}hf_{n+6}$$
$$-\frac{1.1\times10^{13}}{2.1\times10^{12}}hf(Y_0);\ C_9=\frac{5.8\times10^{17}}{1.0\times10^{21}},\ q_1=8,$$

$$Y_2=y_{n+5}-\frac{2.2\times10^{10}}{2.3\times10^{13}}hf_n+\frac{2.6\times10^{10}}{3.2\times10^{12}}hf_{n+1}-\frac{5.1\times10^{11}}{1.5\times10^{13}}hf_{n+2}+\frac{4.8\times10^{10}}{5.7\times10^{11}}hf_{n+3}-\frac{8.3\times10^{9}}{5.2\times10^{10}}hf_{n+4}$$
$$+\frac{1.1\times10^{12}}{1.9\times10^{12}}hf_{n+5}-\frac{2.7\times10^{11}}{1.2\times10^{11}}hf_{n+6}+\frac{4.3\times10^{10}}{1.6\times10^{10}}hf(Y_1);\ C_9=\frac{5.4\times10^{11}}{9.9\times10^{14}},\ q_2=8,$$

$$Y_3=y_{n+5}-\frac{7.8\times10^{7}}{9.1\times10^{10}}hf_n+\frac{6.1\times10^{8}}{8.0\times10^{10}}hf_{n+1}-\frac{6.2\times10^{7}}{2.0\times10^{9}}hf_{n+2}+\frac{1.7\times10^{10}}{2.1\times10^{11}}hf_{n+3}-\frac{9.5\times10^{9}}{6.2\times10^{10}}hf_{n+4}+\frac{2.8\times10^{9}}{5.0\times10^{9}}hf_{n+5}-\frac{1.7\times10^{10}}{1.8\times10^{10}}hf_{n+6}$$
$$+\frac{1.2\times10^{7}}{8.6\times10^{6}}hf(Y_2);C_9=\frac{2.8\times10^{11}}{5.6\times10^{14}},\ \ q_3=8,$$

$$Y_4=y_{n+5}-\frac{1.2\times10^{6}}{1.7\times10^{9}}hf_n+\frac{1.5\times10^{6}}{2.4\times10^{8}}hf_{n+1}-\frac{2.8\times10^{7}}{1.1\times10^{9}}hf_{n+2}+\frac{1.3\times10^{7}}{2.1\times10^{8}}hf_{n+3}-\frac{4.7\times10^{6}}{3.7\times10^{7}}hf_{n+4}+\frac{1.9\times10^{6}}{3.7\times10^{7}}hf_{n+5}-\frac{2.5\times10^{6}}{7.3\times10^{6}}hf_{n+6}$$
$$+\frac{9.9\times10^{6}}{1.5\times10^{7}}hf(Y_3);\ C_9=\frac{3.7\times10^{7}}{9.4\times10^{10}},\ \ q_4=8,$$

$$Y_5=y_{n+5}-\frac{1.3\times10^{4}}{3.6\times10^{7}}hf_n+\frac{5.4\times10^{3}}{1.6\times10^{6}}hf_{n+1}-\frac{1.2\times10^{4}}{8.6\times10^{5}}hf_{n+2}+\frac{6.8\times10^{4}}{1.9\times10^{6}}hf_{n+3}-\frac{6.3\times10^{4}}{8.6\times10^{5}}hf_{n+4}+\frac{1.7\times10^{5}}{4.5\times10^{5}}hf_{n+5}$$
$$-\frac{5.1\times10^{5}}{7.7\times10^{6}}hf_{n+6}+\frac{6.3\times10^{4}}{2.8\times10^{5}}hf(Y_4)\ ;\ C_9=\frac{3.0\times10^{6}}{1.5\times10^{10}},\ \ q_5=8,$$

$$y_{n+6}=y_{n+5}+\frac{-453-533a}{36960(-6+a)}hf_n+\frac{8304-9619a}{90720(-6+a)}hf_{n+1}+\frac{9(-263+298a)}{7840(-6+a)}hf_{n+2}+\frac{984-1079a}{1680(-6+a)}hf_{n+3}+\frac{24487}{30240}+\frac{915}{224(-6+a)}hf_{n+4}$$
$$+\frac{16(19087+\frac{49410}{-6+a})}{218295}hf(Y_5)-\frac{183}{224(-6+a)}h(f_{n+6}+a\,f_{n+5});\ C_8=\frac{-1177}{947520},\ \ p=7 \quad (11)$$

The stability region of the formula in (2) depends on the choice of the value of the parameter “a”. which has the function of smoothing the method for a better stability properties.

Applying the PDNHLM method in (2) to the test equation $y^{/}=\lambda y,\ \mathrm{Re}(\lambda)<0,$ gives the stability polynomial

$$\pi_s(r,z)=w^k-w^{k-1}-z\sum_{j=0}^{k-2}\delta_j w^j-z\theta_{s-1}\left(R_s(w,z)\right)-z\gamma_s\left(w^k+a\,w^{k-1}\right),\quad z=\lambda h, \quad (12)$$

where

$$R_s(w,z)=w^{k-1}+z\sum_{j=0}^{k}\varphi_j w^j+z\rho_{s-2}\left(R_{s-1}(w,z)\left(R_{s-2}(w,z)\left(R_{s-3}(w,z)\left(R_{s-4}(w,z)\left(R_{s-5}(w,z)\right)\right)\right)\right)\right). \quad (13)$$

The behavior of the numerical solution of the PDNHM method will depend on the resulting roots ($w_j(z)$) of $\pi(r,z)=0$.

Adopting the definitions in [10], we have: $R_0=\left\{\pi(w,0)=0 \middle| \max_{1\le j\le k}|w_j(z)|\le 1,\ j=1,2,\ldots,k\right\}$ is called the zero-stability of the PDNHM methods in (2).

- $R_a=\left\{\pi(w,z)=0 \middle| \max_{1\le j\le k}|w_j(z)|\le 1,\ j=1,2,\ldots,k\right\}$ is called the absolute stability of the PDNHM method in (2).
- The PDNHM methods in (2) with stability polynomial $\pi(w,z)$ *is A*(α)-stable if $|\pi(w,z)|\le 1$ for all $z\in S(\alpha)$, $\alpha\in(0,\pi)$, where α is an angle of absolute stability.
- $R_A=\{|\pi(w,z)|\le 1;\ R(z)\le 0\}$ is called *A*-stability of the PDNHM method (2).

The stability plots of the PDNHLM methods in (2) for $k\le 6$ are given in Fig. 1. For

Table 1: Shows the values of "a" and the stability properties of the method in (2)

K	a	Stability Properties
1	½	A-Stable (-∞, 0)
2	¼	A(α)- stable (α = 89.9^0)
3	½	A(α)- stable (α = 47.1^0)
4	½	A(α)- stable (α = 82.5^0)
5	½	A(α)-Stable (α = 61.9^0)
6	¼	A(α)-Stable (α = 58.2^0)

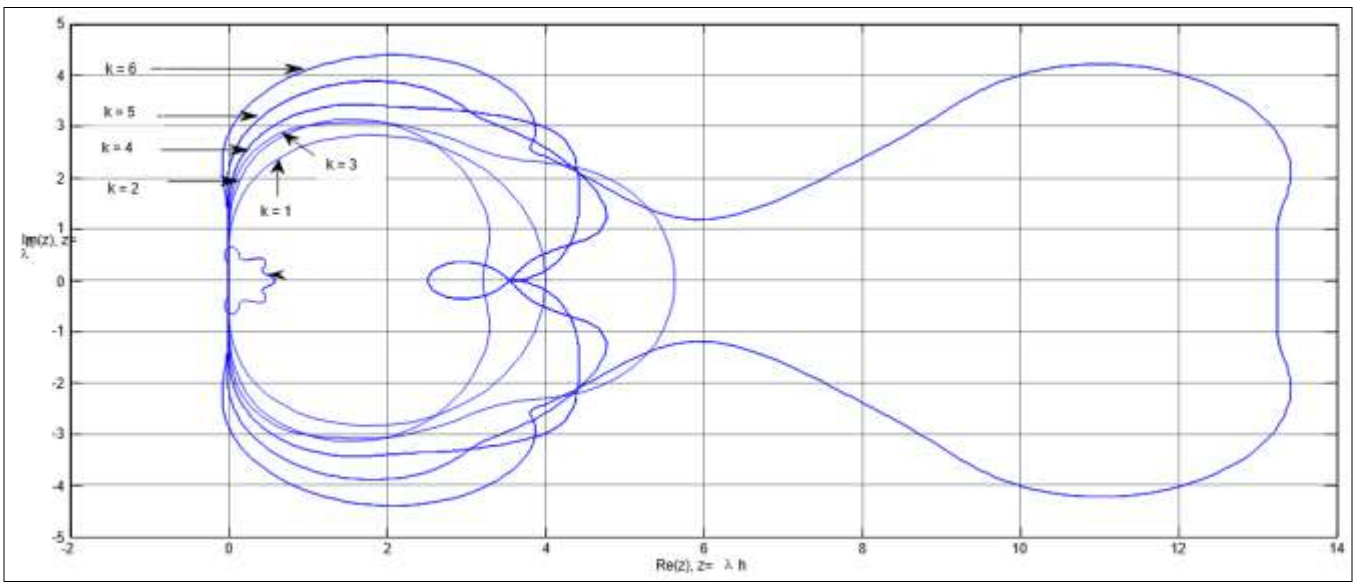

Fig 1: The stability regions (exterior of the closed curves) for the k-step PDNHM method in (2); $k\le 6$.

5. Numerical Experiments

In this section, we implement the method by applying the scheme in (6) for *k=1* to selected standard test stiff initial value problems (IVPs). Our aim is to see how our methods compared with the A(α) – stable LMM [20] and SDLMM [8].

In order to show the competence and superiority of the new methods compared with the well-known methods in the scientific literature, we select the following works for comparison

- PDNHLMM(k_1):the one-step-one-stage method of order 3 derived in (6) of this paper;
- A(α) – stable LMM of order 8; [20]
- SDLMM: the method of order three derived in [8].

In implementing the method, we solve the following standard stiff initial value problems (IVPs)

Problem 1: Nonlinear chemical problem discussed in [8] and [19].

$$y' = \begin{bmatrix} -0.1 & -199.9 \\ -200 & 0 \end{bmatrix} y, \quad y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad h = 0.0001, \quad x = [0,\ 1],$$

$$\textit{Exact solution}: \ y_1(x) = \exp^{-0.1x} + \exp^{-200x} \quad and \quad y_2(x) = \exp^{-200x}$$

Problem 2: Linear and non linear problem discussed in [8] and [25].

$$y' = \begin{bmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{bmatrix} y, \quad y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad h = 0.001, \quad x = [0,\ 1]$$

$$\textit{Exact solution}: \ y(x) = [\exp^{-0.1x}, \exp^{-10x}, \exp^{-100x}, \exp^{-1000x}]^T$$

In solving the IVPs above, the implicitness in the methods has been resolved using the Newton Raphson iterative scheme as suggested by [8] and [9].

$$y_{n+k}^{[s+1]} = y_{n+k}^{[s]} - (F'(y_{n+k}^{[s]}))^{-1} F(y_{n+k}^{[s]}), \quad s = 0, 1, \ldots \tag{14}$$

where $F'(y_{n+k}^{[s]})$ is the Jacobian matrix. While an explicit one-step method $y_{n+1}^{[0]} = y_n + \frac{h}{2}\left(f_{n-1} + f_n\right)$. (15)

in [26] is used to generate the starting value for the schemes in (16)

Table 2: Results for Problem 1 for comparison

x	PDNHLMM y_n	SDLMM (Enright) y_n	Exact Solution $y_{(x)}$	PDNHLMM Error = $\lvert y_{(x)} - y_n \rvert$	SDLMM Error = $\lvert y_{(x)} - y_n \rvert$
0.2	$4.2772760718 \times 10^{-18}$	4.313167705	$4.3341767041 \times 10^{-18}$	$5.6900632264 \times 10^{-20}$	$1.4858821195 \times 10^{-7}$
0.4	$1.8295146962 \times 10^{-35}$	1.839114569	$1.8413118044 \times 10^{-35}$	$1.1797108180 \times 10^{-37}$	$1.6070167507 \times 10^{-7}$
0.6	$7.8253635434 \times 10^{-53}$	7.822634118	$7.8225448396 \times 10^{-53}$	$2.8187037014 \times 10^{-56}$	$1.5004672493 \times 10^{-7}$
0.8	$3.3471343363 \times 10^{-70}$	3.333197363	$3.2329416577 \times 10^{-70}$	$2.3840170623 \times 10^{-72}$	$1.3658646103 \times 10^{-7}$
1.0	$2.0681544656 \times 10^{-9}$	2.093175625	$2.1027916876 \times 10^{-10}$	$3.4637222005 \times 10^{-11}$	$1.2373236302e \times 10^{-7}$

Table 3: Results for Problem 2 for comparison

x	PDNHLMM y_n	A(α)-Stable LMM (Okuonghae & Ikhile) y_n	Exact Solution $y_{(x)}$	PDNHLMM Error = $\lvert y_{(x)} - y_n \rvert$	A(α)-Stable LMM Error = $\lvert y_{(x)} - y_n \rvert$
0.2	$2.2569438502 \times 10^{-9}$	2.119057820	$2.2779270412 \times 10^{-9}$	$2.0983190958 \times 10^{-11}$	$1.9920072216 \times 10^{-16}$
0.4	$5.0959769196 \times 10^{-18}$	5.131579386	$4.6951575726 \times 10^{-18}$	$4.0081934701 \times 10^{-19}$	$5.0021074282 \times 10^{-16}$
0.6	$1.1506259122 \times 10^{-26}$	1.151172254	$9.6774410387 \times 10^{-27}$	$1.8288180836 \times 10^{-27}$	$9.6663711108 \times 10^{-16}$
0.8	$2.5980101770 \times 10^{-35}$	2.596000997	$1.9946692652 \times 10^{-35}$	$6.0334091172 \times 10^{-36}$	$4.7122272259 \times 10^{-16}$
1.0	$5.8660741151 \times 10^{-44}$	5.857155698	$4.1113197817 \times 10^{-44}$	$1.7547543333 \times 10^{-44}$	$3.9931006410 \times 10^{-16}$

The numerical results in Tables 2 and 3 shows that the PDNHLMM for k =1 in methods (6) compare favorably with the SDLMM [8] and A(α)-stable LMM [20] in the literature in the solution of stiff IVPs in ODEs.

Conclusion

This paper proposes and studies the parameter dependent nested hybrid linear multi-step methods (PDNHLMM) for the numerical solution of the IVPs (1). We derived the order conditions for PDNHLM methods via Taylor's series expansion; based on the order conditions presented in this paper, hybrid methods for step number $k \leq 6$ were derived. The results of the numerical experiments confirm that the results of our new methods are more accurate than the high quality methods in the scientific literature.

Following this work, there are several interesting modifications to the PDNHLM methods in (2). For example, a classical modification of the PDNHLM methods can leads to the well-known GLMs [11] of higher order than the existing methods in the literature.

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